2021

MATHEMATICS

Paper - MATH 101

(Abstract Algebra)

(CBCS)

Full Marks : 40 Time : Two Hours

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meanings.

Answer any ***four*** questions of the following.

1. (a) Define conjugate element of a group. Define a relation $\sim$ on a group G by $b \sim a$ if and only if b is conjugate to a. Show that $\sim$ is an equivalence relation in G.

   (b) Let G be a group. Show that $C(a) = \{a\}$ if and only if $a \in Z(G)$.

   [(2+4)+4]

2. (a) Show that the external direct product of two cyclic groups each of order 2 is the Klein 4 group.

   (b) Let $G = H \times K$ be the direct product of H and K. If H and K both are abelian, then show that G is abelian.

   (c) Let $G = H \times K$ be the direct product of H and K. Show that the mapping $p_H : G \to H$ defined by $p_H(h, k) = h$ is a group homomorphism. Find Ker p_H. **[4+2+(2+2)]**

3. (a) Find all Sylow 2-subgroups of the group S_3.

   (b) Let a group G act on a set S and $s_0 \in S$. Show that $\text{Stab}(s_0)$ is a subgroup of G.

   (c) Examine whether the group G of order 42 is simple or not. **[2+4+4]**

4. (a) Let G be a group and x be a fixed element of G. Show that $i_x : G \to G$ defined by $i_x(a) = xax^{-1}$ is an automorphism.

   (b) Define commutator subgroup G' of a group G. If H is any normal subgroup of G such that G/H is abelian, then show that $G' \subseteq H$. **[5+(1+4)]**

5. (a) Let R be a commutative ring with 1. Show that

      (i) $Ra = \{xa : x \in R\}$ is an ideal in R.

      (ii) if R is a simple ring, then R is a field.

   (b) Let P be an ideal in a ring R. Show that P is a prime ideal if and only if R/P is an integral domain. **[(2+3)+5]**

6. (a) Define the norm function on the ring $R = \{a+b\sqrt{-5} : a, b \in \mathbb{Z}\}$. Examine whether $1 + 2\sqrt{-5}$ is a prime element or not in the ring R.

   (b) Let R be an integral domain. If $f(x), g(x) \in R[x]$, then show that $\deg(fg) = \deg(f) + \deg(g)$.

   (c) Let $R = C[0, 1]$ be the ring of real valued continuous functions on $[0, 1]$ and R' be the field of real numbers. Show that the mapping $f : R \to R'$ defined by $f(x) = x(\frac{1}{2})$, $x \in R$ is a ring homomorphism. **[(1+4)+3+2]**
